Classical nonlinear response of a chaotic system. II. Langevin dynamics and spectral decomposition

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The spectrum of a strongly chaotic system consists of discrete complex Ruelle-Pollicott (RP) resonances. We interpret the RP resonances as eigenstates and eigenvalues of the Fokker-Planck operator obtained by adding an infinitesimal diffusion term to the first-order Liouville operator. We demonstrate how the deterministic expression for the linear response is reproduced in the limit of vanishing noise. For the second-order response function we establish an equivalence of the spectral decomposition in the limit of vanishing noise and the long-time asymptotic expansion in the deterministic case.

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I. INTRODUCTION

This paper is the second part of the series that presents our study of the classical nonlinear response in chaotic systems. In our preceding paper [1], we demonstrated the convergence of the second-order response function for strongly chaotic systems. As an example of strong chaos, we considered free motion on a compact surface of constant negative curvature. The model has been studied for more than a century and served as a prototype for quantum chaos [2,3]. The motion on a surface of genus 2 is in particular known as a Hadamard billiard. Using the dynamical symmetry (DS) of the system, we found analytical expressions for the response functions. The long-time asymptotic expansion of the second-order response function turned out to have the form of the double decomposition in the resonances that appear in the linear response.

In the present work, we find the decomposition of the linear and nonlinear response in Ruelle-Pollicott resonances for this classical strongly chaotic model by introducing weak Langevin noise. Spectral decompositions for nonlinear response functions of quantum systems are conceptually straightforward, being expressed via the stationary states of the system [4]. These spectral decompositions, which are often referred to as Bloembergen's expressions, allow one to relate resonances in the spectroscopic measurement data to transitions between stationary states. A natural question arises in the context of the approach to the classical limit: What would be the classical counterpart of the quantum Liouville space spectral decomposition for the linear and nonlinear response functions? In quantum mechanics the system state can be described by a density matrix, whereas the evolution is governed by the quantum Liouville operator. One can try to reach the classical limit by replacing the quantum density matrix by the classical phase-space distribution using the Wigner transformation. The quantum Liouville operator should be naturally replaced by its classical counterpart.

This conceptually straightforward approach, however, faces certain major difficulties. Quantum Liouville operators are second-order elliptic operators, whose eigenstates belong to certain natural Hilbert spaces. In particular, in the case of compact (restricted) coordinate spaces the corresponding spectra are discrete, and finding spectral decompositions does not face conceptual difficulties, at least on the physical level of rigor. The situation with the classical limit is much more involved since the classical Liouville operator is a first-order nonelliptic operator that describes the propagation of phase-space distributions along classical trajectories. Any nonperiodic classical trajectory generates a family of eigenstates of the classical Liouville operator, concentrated on the trajectory with all possible eigenvalues. Such a spectrum contains no useful information on the system relaxation.

The spectrum of a strongly chaotic system consists of Ruelle-Pollicott (RP) resonances [5–7]. Their nonzero real part is responsible for the exponential decay of correlations. An individual RP resonance is not directly related to any particular periodic motion, although its position can be expressed in terms of all periodic orbits in a very collective way through the dynamical ζ function. These collective resonances in chaotic systems should be distinguished from the signatures of stable periodic motions [1,8].

One of the ways to reproduce the RP resonances as eigenvalues of the classical Liouville operator is based on the appropriate simultaneous definition of the functional space where the operator acts. These functional Hilbert spaces, referred to as rigged spaces and usually chosen on a case-by-case basis, are very different from "standard" Hilbert spaces involved in spectral decompositions of quantum operators [9].

To avoid these difficulties we follow a more physical approach [10]. The approach is based on introducing weak Langevin noise, followed by considering the limit of vanishing noise. This approach has several major advantages. First of all, it describes a realistic situation since any system is at least weakly coupled to some environment, and in many cases the system-bath coupling can be described on the level of Langevin noise. Second, this approach does not have the problem of choosing the appropriate Hilbert space. The stochastic Langevin processes can be described by adding a diffusion operator $-\kappa\hat{D}$ to the Liouville operator $\hat{L}$, which results in the Fokker-Planck operator $\hat{\mathcal{L}}=-\kappa\hat{D}+\hat{L}$, where κ is a diffusion coefficient. The Fokker-Planck operator $\hat{\mathcal{L}}$ is a

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second-order elliptic operator, and its spectral decomposition is free of the aforementioned difficulties that arise in the case of the classical Liouville operator. The RP resonances are obtained from the spectrum of the Fokker-Planck operator by taking the limit $\kappa \to 0$. This allows one to interpret introduction of infinitesimal noise as a regularization procedure similar to coarse graining.

The dependence of the eigenfunctions of the Fokker-Planck operator $\hat{\mathcal{L}}$ on the noise strength κ is nonanalytical. In the limit $\kappa \to 0$ of vanishing noise strength the smooth eigenfunctions turn into generalized functions (distributions) that span the rigged Hilbert space. The calculation of the linear response function in terms of the generalized functions does not pose a problem, since the expression involves an expansion of the smooth dipole distribution over generalized functions. In contrast, the calculation of nonlinear response functions turns out to be much more complicated since one needs to define expansions of the generalized functions over similar generalized functions. This problem can be avoided by performing all calculations for weak, yet finite, noise, which is followed by applying the limit $\kappa \to 0$ to the final expressions. The spectral decomposition in this case is conceptually straightforward. We further demonstrate that the decomposition coefficients converge in the noiseless limit $\kappa \to 0$, and the resulting spectral decompositions reproduce the asymptotic expansions for the purely classical response functions computed in Ref. [1]. To the best of our knowledge, this is the first calculation of the RP resonances as well as the spectral decomposition of the linear and second-order response functions, by using noise regularization.

Our paper is organized as follows. In Sec. II we review the geometry and dynamical symmetry of free motion on a compact surface of constant negative curvature. We regularize the dynamics by adding noise to the Liouville operator and present a statistical description of the response in Sec. III. General forms of the spectral decompositions are derived in Sec. IV. The eigenstates of the Fokker-Planck operator are found in the limit of weak noise in Sec. V. We obtain explicit forms of the spectral decompositions for the linear response in Sec. VI and for the second-order response in Sec. VII.

II. MOTION ON A COMPACT SURFACE OF CONSTANT NEGATIVE CURVATURE

Following Ref. [1], we consider free motion on a two-dimensional (2D) compact surface M^2 of constant negative curvature (Gaussian curvature). This strongly chaotic system is described by the classical free-particle Hamiltonian

$$H(\boldsymbol{x},\zeta) = \frac{1}{2m} g^{ik} p_i p_k = \frac{\zeta^2}{2m}, \tag{1}$$

which depends on the absolute value ζ of the momentum $\boldsymbol{p}$ only. The curvature of the 2D configuration space is expressed in terms of the metric tensor g^{ik}. The Hamiltonian classical dynamics preserves the smooth compact 3D manifold M^3, which corresponds to a fixed value of energy. Points $\boldsymbol{x} \in M^3$ of the reduced phase space are described by two coordinates, $\boldsymbol{r} \in M^2$ and the momentum direction angle θ. Hereafter, we will use dimensionless units so that the mass $m=1$ and the curvature $K=-1$. Despite the complexity of the flow due to its chaotic nature, its strong dynamical symmetry enables an analytical treatment. In the preceding paper [1], we presented a detailed review of this model. Below we will provide some basic facts to maintain continuity of exposition.

The geodesic flow $\dot{\boldsymbol{\eta}} = \hat{L}\boldsymbol{\eta}$, where $\boldsymbol{\eta} = (\boldsymbol{r},\boldsymbol{p}) = (\boldsymbol{x},\zeta)$ denotes a point in phase space, is generated by the Liouville operator $\hat{L}$ defined as a Poisson bracket with the free-particle Hamiltonian,

$$\hat{L} \cdots = \{H, \ldots\}. \tag{2}$$

We identify a vector field with the corresponding first-order operator of the derivative in the vector field direction. We denote by σ_1 the vector field that determines the phase-space velocity: $\hat{L} = \zeta\sigma_1$. We also introduce a second vector field $\sigma_z = \partial/\partial\theta$ in the tangent space. In the case of constant (unit) negative curvature the vector fields σ_1, $\sigma_2 = [\sigma_1, \sigma_z]$, and σ_z form the Lie algebra so(2,1), since $[\sigma_1, \sigma_2] = \sigma_z$ and $[\sigma_2, \sigma_z] = -\sigma_1$. The group SO(2,1) action in the reduced phase space M^3 is obtained by integrating the so(2,1) algebra action. Dynamical symmetry with respect to the action of the group A_2 means that σ_1 gives the flow direction, whereas the stable and unstable directions of our hyperbolic flow are determined by $\sigma_z \mp \sigma_2$. In addition, the Poisson bracket of two functions, $f(\boldsymbol{x},\zeta)$ and $g(\boldsymbol{x},\zeta)$, reads

$$\{f,g\} = \frac{\partial f}{\partial \zeta}(\sigma_1 g) - (\sigma_1 f)\frac{\partial g}{\partial \zeta} + \frac{1}{\zeta}[(\sigma_2 f)(\sigma_z g) - (\sigma_z f)(\sigma_2 g)]. \tag{3}$$

The algebra generators σ_l, $l = 1, 2, z$, are anti-Hermitian (i.e., $\sigma_l^\dagger = -\sigma_l$) with respect to the natural scalar product $(\varphi,\psi) = \int d\boldsymbol{x}\ \varphi^*(\boldsymbol{x})\psi(\boldsymbol{x})$.

The space $\mathcal{H}$ of smooth functions in M^3 constitutes a representation of G, which turns out to be unitary (see Refs. [11–13]).Therefore, $\mathcal{H}$ can be decomposed into a direct sum of irreducible representations of G. Stated differently, any distribution in the reduced phase space M^3 can be decomposed into irreducible representations. DS implies that the distributions in different representations evolve independently. We focus on the principal series representations of SO(2,1) since only these provide experimentally interesting contributions to the linear and second-order responses [1]. The principal series representation $\mathcal{H}_s$ is labeled by an imaginary number s, Im $s > 0$. The eigenfunctions $\psi_k(\boldsymbol{x};s)$ of the momentum rotation operator σ_z, hereafter also referred to as angular harmonics, form a convenient basis set in the irreducible representation $\mathcal{H}_s$. The functions $\psi_0(\boldsymbol{x};s)$ do not depend on the momentum direction and can be viewed as eigenfunctions of the Laplacian on our compact Riemann surface M^2 [1,13,14]. The Laplacian eigenvalues provide a set of numbers $s \in \mathrm{Spec}(M^2)$ according to the equation $\nabla^2\psi_0(\boldsymbol{x};s) = (s^2 - 1/4)\psi_0(\boldsymbol{x};s)$. The angular harmonics have the following properties:

$$\sigma_z \psi_k(\boldsymbol{x};s) = ik\psi_k(\boldsymbol{x};s),$$

$$\sigma_\pm \psi_k(\boldsymbol{x};s) = \left(\pm k + \frac{1}{2} - s\right)\psi_{k\pm 1}(\boldsymbol{x};s). \tag{4}$$

The description of unitary representations of G [11,12] allows us to identify the functions $\psi_k(\boldsymbol{x};s)$ in reduced phase space with the corresponding functions on the circle $\Psi_k(u)$ for any $s \in \mathrm{Spec}(M^2)$. In the case of the principal series $\mathrm{Re}\, s=0$, $\mathrm{Im}\, s>0$, and $\Psi_k(u)$ form an orthogonal normalized basis set with the natural scalar product. Therefore, the normalized functions $\psi_k(\boldsymbol{x};s)$ are naturally associated with $\Psi_k(u)=\exp(iku)$. We have the following representation of the algebra generators on the circle:

$$\sigma_z = \frac{d}{du}, \quad \sigma_1 = \sin u \frac{d}{du} + \frac{1-2s}{2}\cos u. \tag{5}$$

The response functions determine the expansion of an observable in powers of the driving field. The coupling to the driving field is characterized by a dipole distribution (polarization) $f(\boldsymbol{r})$, and the observable is chosen to be the phase-space average of $f(\boldsymbol{r})$ [i.e., the convolution of $f(\boldsymbol{r})$ with the phase-space density ρ]. General expressions for classical response functions are given in Eqs. (10) and (11) of Ref. [1], derived from the Liouville equation that governs the phase-space density.

Using the dynamical symmetry of the problem, one can calculate the response functions. We express the dipole $f(\boldsymbol{r})$ in terms of the zero harmonics of irreducible representations:

$$f = \sum_{s \in \mathrm{Spec}_0(M^2)} B_s \psi_0(\boldsymbol{x};s), \tag{6}$$

where only the principal series contributions are retained. Typically the dipole is represented by a smooth distribution, and, therefore, only a small number N_f of representations are relevant in the expansion. For the sake of clarity in the presentation, we focus on the case $N_f=1$.

In the following section we consider the response in the stochastic version of this strongly chaotic model.

III. LANGEVIN DYNAMICS AND FOKKER-PLANCK EQUATION

Phase-space trajectories found from the Hamilton equations of motion are invariant with respect to time reversal. This property is easily observed in the behavior of integrable systems. That is, any integrable dynamics can be represented by a set of quasiperiodic motions, which implies reversibility and recurrence in the values of physical observables. The behavior of chaotic systems is quite different. Although described by the same Hamiltonian formalism, which possesses time-reversal symmetry, chaotic systems obviously exhibit irreversible features such as relaxation phenomena.

This apparent paradox and its solution are well known in statistical mechanics. In the course of evolution of a chaotic system, more and more fine features develop in the phase-space distribution. The distribution width decreases exponentially along the stable directions. At the same time, physical quantities are represented by smooth functions in phase space. Therefore, fine features of the distribution are actually irrelevant for the smooth observables. In particular, this results in the exponential damping of the linear and nonlinear response functions in strongly chaotic systems [1,6,15].

We see that only coarse-grained properties of the phase-space density remain relevant. A physically useful and meaningful definition of the evolution operator which relates initial and final distributions must rely on some kind of regularization that eliminates unnecessary distracting details. Coarse graining can be introduced either directly in phase space or, alternatively, in the functional space of phase-space distributions [16–20]. Sometimes coarse graining is inevitable, as in computer simulations [20].

The finest scale of classical dynamics is limited by the onset of quantum effects. One well-known example using this idea is the semiclassical calculation of the entropy. Quantum effects were also shown to remove unphysical power law divergences in the nonlinear response functions of integrable systems [21,22]. It has been demonstrated that the limit $\hbar \to 0$ should be taken after calculating the long-time asymptotics. This property resembles the characteristic feature of our approach to the second-order response in a chaotic system, as presented below.

In addition, real physical systems are never isolated. The influence of an environment shows up as noise or random forces at the level of the equations of motion, and as unobservable degrees of freedom and diffusive behavior of relevant variables in the statistical description. Although in many cases interaction with the environment is sufficiently weak and can be neglected, it is often utilized as a convenient and physically meaningful way to perform calculations.

The effect of the diffusion in the chaotic system is to regularize the long-time dynamics and to introduce irreversible leveling of gradients in the stable direction. We regularize the operator generating the density evolution (the Liouville operator) by adding a small diffusive term in the form of the second-order differential operator,

$$\hat{\mathcal{L}} = \sigma_1 - \frac{\kappa}{2}\sigma_z^2. \tag{7}$$

This Fokker-Planck operator $\hat{\mathcal{L}}$ defines evolution of the phase-space density in the stochastic case. Since σ_z generates rotations of the momentum, $\kappa/2>0$ has the meaning of the angular diffusion coefficient. The factor 1/2 is chosen for reasons of convenience. We will be interested in the case $\kappa \ll 1$. As we will see, the limit $\kappa \to 0$ does exist in the spectral decomposition; hence the factor 1/2 does not play any role.

While we have chosen a specific type of noise to simplify the following calculations, we should note that, except for some special situations, it is irrelevant because of the mixing property of chaos (see the discussion in Sec. VIII). The statistical description with the Fokker-Planck operator (7) is equivalent to the description in terms of the Langevin equations for the phase-space variables. A random force which tends to change the momentum direction while keeping the energy constant can be formally introduced in a stochastic Liouville operator

$$\hat{L}_{st}(t) = \zeta[\sigma_1 + \gamma(t)\sigma_z], \tag{8}$$

where $\gamma(t)$ is a random Gaussian Markov process with zero mean. The noise intensity is determined by κ, and the two-point correlation function reads

$$\langle \gamma(t)\gamma(t')\rangle = \kappa\delta(t-t'). \tag{9}$$

A random force exerted on any of the phase-space variables quickly affects other variables if they are not fixed by conservation laws. Thus, in the reduced phase space represented by the shell of constant energy, the mixing leads to the fast and irregular randomization of the position variables $\boldsymbol{r}$.

The addition of the diffusion term with the second derivative defines the spectrum of the resulting operator in the space of smooth functions. The eigenfunctions become regular, differentiable functions. As we will see, they are still singular at $\kappa=0$, and in the limit $\kappa\to 0$ they turn into generalized functions, which is natural in the case of a small parameter in front of the highest derivative.

We conclude this section by providing expressions for the response functions. In the stochastic case, the Liouville equation, describing the phase-space density evolution in a system perturbed by an external field $\mathcal{E}$, is replaced by the Fokker-Planck equation

$$(\partial_t + \zeta\hat{\mathcal{L}})\rho = \mathcal{E}(t)\{f,\rho\}. \tag{10}$$

The evolution operator (Perron-Frobenius operator) in the unperturbed system can be written symbolically as $e^{-\zeta\hat{\mathcal{L}}t}$, in full analogy with the noiseless case. Similarly, the iterative solution of Eq. (10) yields the phase-space density in the form of an expansion in powers of $\mathcal{E}$. The response functions can be obtained from Eqs. (10) and (11) of Ref. [1] by substituting $\hat{L}$ for $\zeta\hat{\mathcal{L}}$:

$$S^{(1)}(t) = \int d\zeta\ \zeta\langle f e^{-\zeta\hat{\mathcal{L}}t}\hat{f}_-\rho_0\rangle, \tag{11}$$

$$S^{(2)}(t_1,t_2) = \int d\zeta\ \zeta\langle f e^{-\zeta\hat{\mathcal{L}}t_2}\hat{f}_- e^{-\zeta\hat{\mathcal{L}}t_1}\hat{f}_-\rho_0\rangle, \tag{12}$$

where the angular brackets stand for the integral over the reduced phase space M^3, and the action of the operator $\hat{f}_-$ on a function $g(\boldsymbol{\eta})$ is defined as the Poisson bracket $\hat{f}_- g = \{f,g\}$.

IV. SPECTRAL DECOMPOSITION OF RESPONSE FUNCTIONS: GENERAL FORMALISM

In the well-known situation of a quantum response, the response functions can be readily represented in the form of spectral decompositions, since the infinitesimal evolution is determined by a Hermitian operator. The conjugation property with respect to a simple scalar product facilitates expansions in the basis of the eigenstates.

A natural scalar product for functions in M^3 allowed us to implement the unitary presentations in 3D space by functions on the circle. The first-order differential operator $\hat{L}$ is anti-Hermitian with respect to this scalar product. However, the second-order diffusion contribution makes the resulting Fokker-Planck operator operator $\hat{\mathcal{L}}$ neither Hermitian nor anti-Hermitian. Namely, its adjoint is

$$\hat{\mathcal{L}}^\dagger = -\sigma_1 - \frac{\kappa}{2}\sigma_z^2. \tag{13}$$

It seems that there is no natural choice of the scalar product that would make our Fokker-Planck operator simply related to its adjoint.

The expressions for the response functions include integrations over the reduced phase space. The convolution $\langle\varphi(\boldsymbol{x})\psi(\boldsymbol{x})\rangle \equiv \int d\boldsymbol{x}\ \varphi(\boldsymbol{x})\psi(\boldsymbol{x})$ of two functions $\varphi(\boldsymbol{x})$ and $\psi(\boldsymbol{x})$ can be written as the natural scalar product, $\langle\varphi\psi\rangle = (\varphi^*,\psi)$.

As shown by construction of the eigenmodes in Sec. V, the operator $\hat{\mathcal{L}}$ is diagonalizable in $\mathcal{H}_s$, and the Jordan-block structures that are possible in a general case do not appear in our spectral decompositions. Therefore, any function $\varphi(\boldsymbol{x};s)$ in $\mathcal{H}_s$ can be expanded in the eigenfunctions $\varphi_\lambda(\boldsymbol{x};s)$ that obey the equation

$$\hat{\mathcal{L}}\varphi_\lambda = \lambda\varphi_\lambda. \tag{14}$$

The expansion coefficients are linear functionals of $\varphi(\boldsymbol{x};s)$ and hence can be represented by scalar products of $\varphi(\boldsymbol{x};s)$ with certain functions denoted by $\tilde{\varphi}_\lambda(\boldsymbol{x};s)$:

$$\varphi(\boldsymbol{x};s) = \sum_\mu \langle\tilde{\varphi}^*_\lambda(\boldsymbol{x};s)\varphi(\boldsymbol{x};s)\rangle\varphi_\lambda(\boldsymbol{x};s). \tag{15}$$

It follows from the expansion of $\varphi_\lambda(\boldsymbol{x};s)$ and $\hat{\mathcal{L}}\varphi_\lambda(\boldsymbol{x};s)$ that the functions $\tilde{\varphi}_\lambda(\boldsymbol{x};s)$ are the eigenfunctions of the operator $\hat{\mathcal{L}}^\dagger$ which satisfy the following properties:

$$\hat{\mathcal{L}}^\dagger\tilde{\varphi}_\lambda(\boldsymbol{x};s) = \lambda^*\tilde{\varphi}_\lambda(\boldsymbol{x};s), \tag{16}$$

$$\langle\tilde{\varphi}^*_\mu(\boldsymbol{x};s)\varphi_\lambda(\boldsymbol{x};s)\rangle = \delta_{\mu\lambda}. \tag{17}$$

We focus our detailed analysis on the simplest case $N_f=1$ of a single irreducible representation contributing to the dipole moment $f(\boldsymbol{r})$ [see Eq. (6)], and below imply $f=\psi_0(\boldsymbol{x},s)$. Generalization to an arbitrary linear combination of $N_f>1$ such terms with different s is straightforward for both response functions under consideration [1].

We can successively apply the expansion (15) to obtain any response or correlation function as a spectral decomposition. Based on Eq. (11) we obtain the following expression for the linear response function:

$$S^{(1)}(t) = \int d\zeta\ \zeta\sum_\lambda e^{-\lambda\zeta t}\langle f\varphi_\lambda\rangle\langle\tilde{\varphi}^*_\lambda\hat{f}_-\rho_0\rangle. \tag{18}$$

We rewrite the expression for the second-order response function in Eq. (12) as

$$S^{(2)}(t_1,t_2) = \int d\zeta\ \zeta\sum_{mn}\sum_{\mu\lambda} e^{-\mu\zeta t_2}\langle f\varphi_\mu\rangle\langle\tilde{\varphi}^*_\mu\psi_n\rangle\langle\psi^*_n\hat{f}_-\psi_m\rangle \times\langle\psi^*_m e^{-\lambda\zeta t_1}\varphi_\lambda\rangle\langle\tilde{\varphi}^*_\lambda\hat{f}_-\rho_0\rangle, \tag{19}$$

introducing two additional expansions in the angular harmonics defined by Eqs. (4). The phase-space integration in the middle angular brackets in Eq. (19) yields a geometric matrix element $\langle \psi_n^* \psi_0 \psi_m \rangle$ studied in our preceding paper [1]. The other angular brackets in Eqs. (18) and (19) correspond to projections of the angular harmonics ψ_n on the eigenfunctions φ_λ of the Fokker-Planck operator and vice versa. Note that $\hat{f}_-$ in the middle angular brackets contains a derivative $\partial/\partial\zeta$ acting on all functions of the momentum to the right of it. This complication occurs due to the absence of integration over ζ in the integral denoted by the angular brackets.

V. EIGENSTATES OF THE FOKKER-PLANCK OPERATOR

As stated earlier, the Fokker-Planck operator (7) is neither Hermitian nor anti-Hermitian with respect to any natural scalar product and, therefore, not necessarily diagonalizable. In what follows, we show that it can actually be diagonalized. We apply the representation on the circle to find its eigenstates and demonstrate that they constitute a basis set.

The eigenfunctions and eigenvalues of $\hat{\mathcal{L}}$ can be found by solving the 1D eigenvalue problem

$$\hat{\mathcal{L}}\Phi_\lambda = \lambda\Phi_\lambda \tag{20}$$

on a circle with the Fokker-Planck operator [see Eqs. (5) and (7)]

$$\hat{\mathcal{L}} = -\frac{\kappa}{2}\frac{d^2}{du^2} + \sin u\frac{d}{du} + \frac{1-2s}{2}\cos u. \tag{21}$$

The first derivative in the operator can be eliminated by redefining the functions

$$\Phi_\lambda(u) = e^{-\cos u/\kappa}\xi_\lambda(u). \tag{22}$$

In terms of the functions $\xi_\lambda(u)$, the eigenvalue problem (20) assumes the form of a stationary Schrödinger equation

$$\hat{\mathcal{H}}\xi_\lambda(u) = \lambda\xi_\lambda(u) \tag{23}$$

with the effective Hamiltonian

$$\hat{\mathcal{H}} = -\frac{\kappa}{2}\frac{\partial^2}{\partial u^2} + \frac{\sin^2 u}{2\kappa} - s\cos u. \tag{24}$$

The Hamiltonian $\hat{\mathcal{H}}$ describes a quantum particle in a complex-valued potential on the circle. The operator is not Hermitian because of the imaginary part of the potential. However, since the Hamiltonian does not contain first derivatives, we can define a symmetric (non-Hermitian) scalar product $V(\xi,\phi)=\int_{-\pi}^{\pi} du\ \xi(u)\phi(u)$ so that the Hamiltonian is self-adjoint with respect to it: $V(\hat{\mathcal{H}}\xi,\phi)=V(\xi,\hat{\mathcal{H}}\phi)$. Eigenfunctions of $\hat{\mathcal{H}}$ corresponding to different eigenvalues are orthogonal and can be normalized:

$$\int_{-\pi}^{\pi} du\ \xi_\lambda(u)\xi_\mu(u) = \delta_{\lambda\mu}. \tag{25}$$

The Hamiltonian $\hat{\mathcal{H}}$ also possesses certain symmetries that simplify the analysis. First, it is invariant with respect to a sign change of u, and therefore all its eigenfunctions are either even or odd functions of u. As we will see, only even eigenstates contribute to the spectral decompositions of response functions. Therefore, we present the detailed expressions only for even eigenstates on the half-circle $0<u<\pi$.

Since the potential contains a nonzero imaginary part, its eigenvalues can be complex. The second symmetry involves the complex conjugation of the Hamiltonian: $\hat{\mathcal{H}}^*(u)=\hat{\mathcal{H}}(u+\pi)$. Consequently, the eigenfunctions corresponding to the complex conjugate eigenvalues are related as

$$\xi_{\lambda^*}(u) = [\xi_\lambda(u+\pi)]^*. \tag{26}$$

Detailed analysis of the Schrödinger equation in the weak noise case $\kappa \ll 1$ is presented in the Appendix. The smallness of the parameter κ allows one to solve the Schrödinger equation using the WKB method. The imaginary part of the potential energy is small compared to its real part. This property supports the use of such terms as the "minimum" of the potential and the "under the barrier" wave function. Since the potential minima become deeper for smaller κ, the eigenfunctions are concentrated near $u=0$ or $u=\pi$. These states do not mix since s is imaginary.

Due to the compactness of the circle, the spectrum of the Fokker-Planck operator is discrete. Its real part is positive and unbounded from above. For such large energies that $\mathrm{Re}\,\lambda \gg \kappa^{-1}$, the spectrum can be adequately approximated by $\lambda_\nu \sim \kappa\nu^2/2$ (with the eigenfunctions $\xi_{\lambda_\nu} \propto e^{i\nu u}$), which corresponds to a free particle on the circle.

In the limit $\kappa \to 0$ even low-energy eigenstates of the first set are concentrated near $u=0$ and have energies

$$\lambda_\nu = \nu - s + \frac{1}{2}, \tag{27}$$

with nonnegative even numbers ν ($\nu=0,2,4,\ldots$). This discrete equidistant spectrum of the Fokker-Planck operator in the noiseless limit $\kappa \to 0$ is a quite fascinating property. Infinitesimal noise regularizes the Liouvillian dynamics to yield a physical spectrum in the space of smooth functions.

The normalized eigenfunctions that correspond to the specified above eigenvalues are concentrated near $u=0$:

$$\xi_{\lambda_\nu}(u) = A_{0,\nu} e^{-u^2/2\kappa} H_\nu\left(\frac{u}{\sqrt{\kappa}}\right), \tag{28}$$

$$A_{0,\nu} = (\pi\kappa)^{-1/4}(2^\nu \nu!)^{-1/2}, \tag{29}$$

where H_ν are Hermite polynomials. The eigenfunctions $\Phi_{\lambda_\nu}(u)=e^{\cos u/\kappa}\xi_{\lambda_\nu}(u)$ are singular at $u=0$ in the limit $\kappa \to 0$, as they should form a basis set of distributions with large gradients in the stable direction.

In the limit $\kappa \to 0$ the eigenvalues in Eq. (27) are determined by the single potential minimum at $u=0$. The form (28) of the eigenfunctions is appropriate in the region where the potential may be approximated by a harmonic well. Outside the region, at $|u| \gtrsim 1$, the values of the eigenfunction are exponentially small and do not influence its normalization.

A more rigorous treatment in the Appendix provides the WKB approximation for the even eigenfunctions $\xi_\nu(u)$ on the whole circle. Under the potential barrier, if $u \gtrsim \kappa^{1/2}$ and

$(\pi-u)\gtrsim\kappa^{1/2}$, the approximate solution at the energy $\lambda_\nu=\nu-s+1/2$ is

$$\xi(u)=A_{2,\nu}2^{s-1/2}\left(\sin\frac{u}{2}\right)^{\nu}\left(\cos\frac{u}{2}\right)^{s-\nu-1/2}e^{\cos u/\kappa}, \quad (30)$$

$$A_{2,\nu}=A_{0,\nu}e^{-1/\kappa}2^{\nu+\lambda}\kappa^{-\nu/2}. \quad (31)$$

In the vicinity of $u=\pi$, at $\pi-u\ll 1$, we obtain:

$$\xi(u)=A_1e^{-(u-\pi)^2/2\kappa}H_{\nu_1}\left(\frac{u-\pi}{\sqrt{\kappa}}\right)+B_1e^{(u-\pi)^2/2\kappa}H_{-\nu_1-1}\left(i\frac{u-\pi}{\sqrt{\kappa}}\right), \quad (32)$$

where

$$\nu_1=\nu-2s=\lambda-\frac{1}{2}-s, \quad (33)$$

$$A_1=-B_1i2^{-\nu_1}\pi^{-1/2}\Gamma(-\nu_1)\cos\frac{\pi\nu_1}{2}, \quad (34)$$

$$B_1=A_{2,\nu}e^{-1/\kappa}2^{\nu_1+\lambda+1}\kappa^{-(\nu_1+1)/2}e^{i\pi(\nu_1+1)/2}. \quad (35)$$

Even low-lying eigenstates of the second set concentrated near $u=\pi$ have eigenenergies $\lambda^*_\nu=\nu+s+1/2$, and their eigenfunctions can be obtained by using Eq. (26), which follows from the symmetry of the Hamiltonian.

Equations (28)–(35) provide a zero-order approximation for the eigenfunction of the Hamiltonian (24) on the circle. In the limit $\kappa\ll 1$ one can distinguish two types of corrections to the eigenvalues and eigenfunctions of low-lying states specified above. The first one is due to the terms neglected in the potential in the vicinity of its minima. Such corrections are important in the treatment of states with $\nu>0$. Corrections of the second type originate from the tunneling through the potential barriers and therefore are exponentially small, proportional to $e^{-2/\kappa}$.

Details of computing the eigenfunctions are presented in the Appendix. It turns out that a straightforward calculation of the higher-mode contributions to the spectral decomposition requires knowledge of higher-order corrections to the eigenfunctions. In the next two sections we present a more elegant approach for which the approximation given by Eqs. (28)–(35) is sufficient.

VI. SPECTRAL DECOMPOSITION FOR THE LINEAR RESPONSE

Spectral decompositions of the response functions in the eigenmodes of the evolution operator are obtained in Eqs. (18) and (19) in the general form. The expressions are not really useful for calculation, since they involve several three-dimensional integrations with functions that are not explicitly known. Nevertheless, the calculation can be carried out by implementing the representation in terms of functions on the circle.

We start with the spectral decomposition of the linear response, and employ the correspondence between the functions φ in M^3 and $\Phi(u)$ on the circle to obtain the following expansion:

$$\Phi(u)=\sum_\lambda\langle\tilde{\varphi}^*_\lambda\varphi\rangle\Phi_\lambda(u). \quad (36)$$

Using the representation $\Phi_\lambda(u)=e^{-\cos u/\kappa}\xi_\lambda(u)$ and the orthonormality of the functions $\xi_\lambda(u)$ with respect to the symmetric scalar product (25), we can specify a rule to find a projection of φ on φ_λ:

$$\langle\tilde{\varphi}^*_\lambda\varphi\rangle=\int_{-\pi}^{\pi}du\ \Phi(u)e^{\cos u/\kappa}\xi_\lambda(u). \quad (37)$$

This implies that $\tilde{\varphi}_\lambda$ is implemented on the circle by the function

$$\tilde{\Phi}_\lambda(u)=2\pi e^{\cos u/\kappa}\xi^*_\lambda(u). \quad (38)$$

Next we introduce the notation

$$R_{\lambda,n}\equiv\langle\tilde{\varphi}^*_\lambda\psi_n\rangle=\int_{-\pi}^{\pi}du\ e^{inu}e^{\cos u/\kappa}\xi_\lambda(u) \quad (39)$$

for the coefficient of the expansion of angular harmonics $\psi_n(\boldsymbol{x};s)$ in the basis of eigenfunctions $\varphi_\lambda(\boldsymbol{x};s)$ of the noisy evolution operator. The expansion of $\varphi_\lambda(\boldsymbol{x};s)$ in the basis of $\psi_n(\boldsymbol{x};s)$ contains coefficients

$$\langle\psi^*_n\varphi_\lambda\rangle=\int_{-\pi}^{\pi}\frac{du}{2\pi}e^{-inu}e^{-\cos u/\kappa}\xi_\lambda(u), \quad (40)$$

which are related to $R_{\lambda,n}$ due to the symmetry property (26),

$$\langle\psi^*_n\varphi_\lambda\rangle=\frac{(-1)^n}{2\pi}(R_{\lambda^*,n})^*. \quad (41)$$

Therefore, the convolution in the first angular brackets in Eq. (18) is equal to $(R_{\lambda^*,0})^*/2\pi$. The action of f_- (represented by the Poisson bracket with f) in the last angular brackets produces the following convolution:

$$\langle\tilde{\varphi}^*_\lambda\sigma_1\psi_0\rangle=\frac{1-2s}{2}\int_{-\pi}^{\pi}du\cos u\ e^{\cos u/\kappa}\xi_\lambda(u), \quad (42)$$

where we have used the representation of σ_1 on the circle given by Eq. (5). After integrating both sides of Eq. (23) (the integration by parts in the left-hand side eliminates the second derivative) and comparing the result with Eq. (42), we find the following relation:

$$\langle\tilde{\varphi}^*_\lambda\sigma_1\psi_0\rangle=\lambda R_{\lambda,0}. \quad (43)$$

We conclude that the linear response function can be recast in a form similar to the time derivative of the two-point correlation function, in agreement with the fluctuation-dissipation theorem (FDT):

$$S^{(1)}(t)=\frac{\partial}{\partial t}\sum_\lambda\int_0^\infty d\zeta\ \mathcal{Q}_{\lambda,0}e^{-\xi\lambda t}\frac{\partial\rho_0}{\partial\zeta}, \quad (44)$$

where we introduced

$$\mathcal{Q}_{\lambda,n} = \frac{(-1)^n}{2\pi} R_{\lambda,0} (R_{\lambda^*,n})^*. \tag{45}$$

Following the approach developed in Ref. [1] for the purely deterministic situation, we introduce matrix elements of the evolution operator between the nth and zeroth harmonics:

$$\mathcal{A}_n(\zeta t;s) = \int d\boldsymbol{x}\ \psi_n^*(\boldsymbol{x};s) e^{-\hat{\mathcal{L}}t} \psi_0(\boldsymbol{x};s). \tag{46}$$

Using the representation on the circle, we will find the spectral decomposition

$$\mathcal{A}_n(\zeta t;s) = \sum_{\lambda} \mathcal{Q}_{\lambda,n} e^{-\zeta\lambda t}, \tag{47}$$

which appears in the response functions of the system with noise.

As shown below, in the noiseless limit $\kappa \to 0$, the spectral decomposition of $\mathcal{A}_n(t;s)$ [Eq. (47)] is equal to the matrix element $A_n(t;s)$ of the deterministic evolution operator, calculated in Ref. [1]. The series (47) and its coefficients $\mathcal{Q}_{\lambda,n}$ play an important role in the calculation of the second-order response function.

We are now in a position to proceed with an explicit calculation of the coefficients in the spectral decomposition of the linear response function. We focus on the first set of modes with the energies $\lambda_\nu = \nu - s + 1/2$, whose eigenfunctions are concentrated in the neighborhood of $u=0$ for $\kappa \ll 1$. The coefficients $\mathcal{Q}_{\lambda,0}$ for the other set of modes, with the energies $\lambda_\nu^* = \nu + s + 1/2$, can then be easily found by employing the symmetry, described by Eq. (26):

$$\mathcal{Q}_{\lambda^*,0} = (\mathcal{Q}_{\lambda,0})^*,$$

which also reflects the fact that the response function is real. Since odd eigenfunctions are orthogonal to the momentum-independent dipole distribution f, which formally means $R_{\lambda,0}=0$, they do not contribute to the spectral decomposition of the response functions.

The first two terms in the spectral decomposition originate from the lowest RP resonances with the energies $\lambda_0 = 1/2 - s$ and $\lambda_0^* = 1/2 + s$. The eigenfunction with $\lambda_0 = 1/2 - s$, concentrated in the vicinity of $u=0$, is given by $\xi_{\lambda_0}(u) = (\pi\kappa)^{-1/4} \exp(-u^2/2\kappa)$. The first integral $R_{\lambda_0,0}$ is calculated without much effort. We notice that the main contribution to the integral comes from $|u| \lesssim \sqrt{\kappa}$. Expanding $\cos u = 1 - u^2/2$ in the exponential and performing a Gaussian integral, we obtain

$$R_{\lambda_0,0} = e^{1/\kappa} A_{0,0}^{-1}, \tag{48}$$

where the normalization factor $A_{0,0}$ is related to κ via Eq. (29).

The situation, however, is more complicated for higher modes with $\nu > 0$. Due to the orthogonality of the eigenfunctions ξ_{λ_0} and ξ_{λ_ν} with $\nu > 0$, the integral vanishes if we employ the lowest-order approximation for the wave functions:

$$\int_{-\pi}^{\pi} du\ e^{\cos u/\kappa} \xi_{\lambda_\nu}(u) \approx A_{0,\nu} e^{1/\kappa} \int_{|u| \lesssim \sqrt{\kappa}} du\ e^{-u^2/\kappa} H_\nu\left(\frac{u}{\sqrt{\kappa}}\right) = 0. \tag{49}$$

This result merely means that we should use more accurate approximations for both the exponential $e^{\cos u/\kappa}$ and the wave function ξ_{λ_ν}. The corrections to the latter are computed by applying the standard quantum mechanical perturbation theory to the Schrödinger equation. The procedure is feasible for a few low modes only, since the calculation for higher modes requires higher orders of perturbation theory.

However, it is possible to overcome this difficulty and find an explicit expression for $R_{\lambda_\nu,0}$ for all ν by using the following trick. We notice that the dominant contribution to the integral

$$\int_{-\pi}^{\pi} du\ e^{\cos u/\kappa} (\cos u - 1)^{\nu/2} \xi_{\lambda_\nu}(u) = 2^{-2\nu} (-1)^{\nu/2} A_{0,\nu}^{-1} e^{1/\kappa}. \tag{50}$$

can be found without the higher-order perturbative calculations outlined above. The dominant contribution to the integral comes from the region $|u| \lesssim \sqrt{\kappa}$, where we can approximate $(\cos u - 1)^{\nu/2} = (-1/2)^{\nu/2} u^\nu$ and expand it in the Hermite polynomials $H_j(u/\sqrt{\kappa})$ with $j = 0, 2, \ldots, \nu$. Only the last term results in a nonzero integral due to the orthogonality of the Hermite polynomials. Further approximations lead to negligible corrections in the limit $\kappa \to 0$. We proceed by introducing

$$P_{j,\lambda} \equiv \int_{-\pi}^{\pi} du\ e^{\cos u/\kappa} (\cos u - 1)^j \xi_\lambda(u). \tag{51}$$

Our goal is to calculate the integral $R_{\lambda_\nu,0} = P_{0,\lambda_\nu}$ for $\kappa \ll 1$. This will be achieved by expressing it in terms of the known integral $P_{\nu/2,\lambda_\nu}$ specified by Eq. (50). Consider $\lambda P_{j,\lambda}$, where we replace $\lambda \xi_\lambda$ by $\hat{\mathcal{H}} \xi_\lambda$ and integrate the term with the second derivative by parts, neglecting higher-order terms in κ. Thus we obtain a recurrence relation

$$P_{j,\lambda} = \frac{j - s + \frac{1}{2}}{\lambda - 2j + s - \frac{1}{2}} P_{j+1,\lambda}, \tag{52}$$

which, being applied $\nu/2$ times and followed by setting $\lambda = \lambda_\nu$ from Eq. (27), yields

$$P_{0,\lambda_\nu} = \frac{2^{-\nu/2} \Gamma\left(\frac{\nu+1-2s}{2}\right)}{(\nu/2)!\, \Gamma\left(\frac{1-2s}{2}\right)} P_{\nu/2,\lambda_\nu}. \tag{53}$$

We finally arrive at the following expression:

$$R_{\lambda_\nu,0} = e^{1/\kappa} \kappa^{\nu/2} A_{0,\nu}^{-1} 2^{-2\nu} \frac{(-1)^{\nu/2} \Gamma\left(\frac{\nu+1-2s}{2}\right)}{(\nu/2)!\, \Gamma\left(\frac{1-2s}{2}\right)}. \tag{54}$$

The second integral $\langle \psi_0 \varphi_{\lambda_\nu} \rangle$ in the linear response is expressed through $(R_{\lambda_\nu^*,0})^*$ [see Eqs. (40) and (41)],

$$(R_{\lambda^*_{\nu},0})^* = \int_{-\pi}^{\pi} du\, e^{-\cos u/\kappa} \xi_{\lambda_\nu}(u). \tag{55}$$

The obtained integral looks more complicated than $R_{\lambda_{\nu},0}$, even for the lowest resonance with $\nu=0$. We first point out the problem and present its straightforward solution for $\nu=0$. Afterward we will apply a more elegant method, similar to the one developed above, which provides a solution for all modes.

As stated earlier, it is sufficient to restrict the analysis of the integral to the interval $0<u<\pi$. Analytical approximate forms for the eigenfunction in three regions near $u=0$, $u=\pi$, and under the barrier are specified in Eqs. (28)–(35). In contrast to the case of $R_{\lambda_{\nu},0}$, the integrand in $(R_{\lambda^*_{\nu},0})^*$ does not depend on u exponentially, and the exponential part of its dependence on κ in the contributions from all three regions amounts to a common factor $\exp(-1/\kappa)$.

The eigenfunction is $\xi_{\lambda_\nu}(u) \propto (\pi-u)^{-\nu-1+2s}$ in the region $\sqrt{\kappa} \ll (\pi-u) \ll 1$, where the approximate forms given by Eqs. (30) and (32) match. This dependence indicates that one has to take into account the contributions of both approximations. We can break up the integral into two parts by introducing an intermediate point $\pi-u_1$. Since for small $\kappa \ll 1$ one can always choose the point $\pi-u_1$ to belong to the matching region (i.e., $\kappa^{1/2} \ll u_1 \ll 1$), the approximate forms (30) and (32) can be used to the left and to the right of the intermediate point, respectively. We intend to show that the sum of two integrals is independent of u_1.

The contribution from the region near $u=\pi$ has the form

$$\begin{aligned}\int_{\pi-u_1}^{\pi} du\, e^{-\cos u/\kappa} \xi_{\lambda}(u) &= e^{1/\kappa}\sqrt{\kappa}\int_0^{u_1/\sqrt{\kappa}} dz [A_1 e^{-z^2} H_{\nu_1}(z) \\ &\quad + B_1 H_{-\nu_1-1}(iz)] \\ &= -B_1 e^{1/\kappa}\sqrt{\kappa} e^{-i\pi\nu_1/2} \\ &\quad \times \frac{H_{-\nu_1}(iu_1/\sqrt{\kappa}) - H_{-\nu_1}(-iu_1/\sqrt{\kappa})}{4\nu_1 \sin(\pi\nu_1/2)}.\end{aligned} \tag{56}$$

Choosing u_1 so that $\kappa^{1/2} \ll u_1 \ll \kappa^{1/4} \ll 1$, we have retained only the first two terms in the expansion of $\cos u$. The integral has been calculated by expressing $e^{-z^2} H_{\nu_1}(z)$ in terms of $H_{-\nu_1-1}(iz)$ and $H_{-\nu_1-1}(-iz)$ (all three functions are solutions of the same second-order differential equation [23]). We have also employed the symmetry of the eigenfunctions and the recurrence relation $\frac{d}{dz} H_{-\nu_1}(iz) = -2i\nu_1 H_{-\nu_1-1}(iz)$ for Hermite functions.

The integration with the under-the-barrier function [specified by Eqs. (30) and (31)] can be safely extended to $u=0$, since the deviation of the approximate integrand from the exact one in a short region of length $\sim\sqrt{\kappa}$ is finite. Then the integration is performed exactly in terms of the hypergeometric function:

$$\begin{aligned}&\int_0^{\pi-u_1} du \left(\sin\frac{u}{2}\right)^{\nu} \left(\cos\frac{u}{2}\right)^{s-\nu-1/2} \\ &= \frac{\Gamma\left(\frac{\nu+1}{2}\right)\Gamma\left(\frac{-\nu_1}{2}\right)}{\Gamma(s+1/2)} + \frac{2\ {}_2F_1\left(1, s+\frac{1}{2}, \frac{2-\nu_1}{2}, \sin^2\frac{u_1}{2}\right)}{\nu_1 \left(\sin\frac{u_1}{2}\right)^{\nu_1}} \left(\cos\frac{u_1}{2}\right)^{\nu+1}.\end{aligned} \tag{57}$$

We further expand Eq. (56) in $\sqrt{\kappa}/u_1 \ll 1$ and Eq. (57) in $u_1 \ll 1$. In the sum of two pieces of the integral (55), the strongest dependence on u_1 cancels out. For the lowest resonance with $\nu=0$ there are no other contributions comparable with the constant term in Eq. (57), which is not the case if $\nu>0$.

As ν increases, the divergence of the underbarrier solution becomes stronger, and one should take into account all u_1-dependent terms with nonpositive real parts of the exponents that appear in Eqs. (56) and (57). For large ν this approach requires higher-order perturbative calculations. One might have argued that, since the corrections to the wave functions have the same structure as the initial approximations for them, all relevant u_1-dependent terms should cancel out, and the corrections to the constant term are small in κ.

However, we can avoid tedious calculations by applying the method developed above. This is achieved by relating $(R_{\lambda^*_{\nu},0})^*$ to the integral that can be calculated by using the principal approximation for the eigenfunction specified in Eqs. (28)–(35). According to the definition (51) of $P_{j,\lambda}$ and due to the symmetry of the wave function [see Eq. (26)], we have $(R_{\lambda^*_{\nu},0})^* = (P_{0,\lambda^*_{\nu}})^*$. The recurrence relation (52) yields

$$(P_{0,\lambda^*_{\nu}})^* = \frac{\Gamma\left(s+\frac{\nu+3}{2}\right)\Gamma\left(s-\frac{\nu}{2}\right)}{(-2)^{1+\nu/2}\Gamma(s+1/2)\Gamma(1+s)} (P_{1+\nu/2,\lambda^*_{\nu}})^*. \tag{58}$$

Next, we consider the integral

$$(P_{1+\nu/2,\lambda^*_{\nu}})^* = (-1)^{1+\nu/2} \int_{-\pi}^{\pi} du\, e^{-\cos u/\kappa} (1+\cos u)^{1+\nu/2} \xi_{\lambda_\nu}(u) \tag{59}$$

and notice that for its calculation it is sufficient to take the underbarrier approximation (30) for ξ_{λ_ν} on the whole half circle. The resulting integral can be easily calculated:

$$(P_{1+\nu/2,\lambda^*_{\nu}})^* = A_{2,\nu}(-1)^{1+\nu/2} 2^{(\nu+2s+3)/2} \frac{\Gamma\left(\frac{1+\nu}{2}\right)\Gamma(1+s)}{\Gamma[(2s+\nu+3)/2]}. \tag{60}$$

The integral converges, and the differences between the approximate and exact integrands in the narrow regions of width $\sim\sqrt{\kappa}$ near $u=0$ and $u=\pi$ are smaller than $\sim A_{2,\nu}\kappa^{\nu/2}$ and $\sim A_{2,\nu}\kappa^{s+1/2}$, respectively. The calculation of the second integral is completed by combining Eqs. (58) and (60) followed by expressing $A_{2,\nu}$ via $A_{0,\nu}$, according to Eq. (31):

$$(R_{\lambda^*_{\nu},0})^* = A_{0,\nu} e^{-1/\kappa} \kappa^{-\nu/2} 2^{1+2\nu} \frac{\Gamma\left(\frac{\nu+1}{2}\right)\Gamma\left(\frac{2s-\nu}{2}\right)}{\Gamma(s+1/2)}. \quad (61)$$

The method is naturally also applicable to the case $\nu=0$, which has been treated earlier, by using a more straightforward approach, with much more effort involved.

Inserting Eqs. (54) and (61) into Eq. (45), we find that the coefficients $\mathcal{Q}_{\lambda_{\nu},0}$ and $\mathcal{Q}_{\lambda^*_{\nu},0}$ have finite limits at $\kappa \to 0$:

$$\lim_{\kappa\to 0} \mathcal{Q}_{\lambda_{\nu},0} = \frac{\Gamma\left(\frac{1+\nu}{2}\right)}{\pi(\nu/2)!} \frac{\Gamma\left(\frac{1-2s+\nu}{2}\right)}{\Gamma\left(1+\frac{\nu-2s}{2}\right)} \cot(\pi s), \quad (62)$$

$$\lim_{\kappa\to 0} \mathcal{Q}_{\lambda^*_{\nu},0} = (\lim_{\kappa\to 0} \mathcal{Q}_{\lambda_{\nu},0})^*.$$

We conclude the section by demonstrating that the spectral decomposition of the linear response in the noiseless limit $\kappa \to 0$ reproduces the asymptotic expansion of its purely deterministic counterpart calculated in Ref. [1]. To that end we show that in the noiseless limit $\mathcal{Q}_{\lambda_{\nu},0}$ and $\mathcal{Q}_{\lambda^*_{\nu},0}$ coincide with the coefficients in the expansion

$$A_n(t;s) = \sum_{\nu=0,2,\ldots} (Q_{\nu,n} e^{st} + \tilde{Q}_{\nu,n} e^{-st}) e^{-(\nu+1/2)t} \quad (63)$$

at $n=0$, where $A_n(t;s)$ is defined as a matrix element of the Liouvillian deterministic evolution:

$$A_n(\zeta t;s) = \int d\boldsymbol{x}\ \psi^*_n(\boldsymbol{x};s) e^{-\hat{L}t} \psi_0(\boldsymbol{x};s). \quad (64)$$

The integral can be calculated by using the representation on the circle (see Refs. [1,2]):

$$A_n(t;s) = \frac{2(e^{-2t}-1)^n \Gamma\left(n+\frac{1}{2}-s\right)}{\sqrt{\pi}\Gamma\left(\frac{1}{2}-s\right)} e^{-t/2} \times \mathrm{Re}\left[\frac{\Gamma(s)e^{st}}{\Gamma\left(n+\frac{1}{2}+s\right)} {}_2F_1\left(n+\frac{1}{2} - s, n+\frac{1}{2}, 1-s, e^{-2t}\right)\right]. \quad (65)$$

The expression is substantially simplified in the relevant for the linear response case $n=0$. Expanding the Gauss hypergeometric function ${}_2F_1$ in $A_0(t;s)$ into the hypergeometric series of e^{-2t} to obtain $Q_{\nu,0}$ and $\tilde{Q}_{\nu,0}$, we see directly that they are indeed reproduced by the noiseless limits ($\kappa \to 0$) of the coefficients $\mathcal{Q}_{\lambda_{\nu},0}$ and $\mathcal{Q}_{\lambda^*_{\nu},0}$ given by Eqs. (62):

$$\mathcal{Q}_{\lambda_{\nu},0} = Q_{\nu,0} \quad \text{and} \quad \mathcal{Q}_{\lambda^*_{\nu},0} = \tilde{Q}_{\nu,0}. \quad (66)$$

VII. SPECTRAL DECOMPOSITION OF THE SECOND-ORDER RESPONSE: NOISE REGULARIZATION

Spectral decomposition of nonlinear response functions in the case of finite noise is conceptually straightforward. The general expression for $S^{(2)}(t_1,t_2)$, decomposed into the eigenmodes of the Fokker-Planck operator $\hat{\mathcal{L}}(\kappa)$ is given by Eq. (19). In this section we demonstrate that in the noiseless limit $\kappa \to 0$ the expansion coefficients converge to the coefficients of the long-time asymptotic expansion of the purely deterministic $\kappa=0$ response that have been derived in the preceding paper [1]. For the sake of simplicity, we focus on the $N_f=1$ case when only one irreducible representation contributes to the dipole, i.e., $f=\psi_0(\boldsymbol{x};s)$. The expression (19) for the second-order response contains four matrix elements calculated in the previous section and another matrix element in the middle angular brackets, which requires a more careful treatment since $\hat{f}_-$ includes the operator $\sigma_1 \psi_0 \partial/\partial\zeta$ acting on all momentum-dependent functions to the right. We first perform the integration over the reduced phase space, represented by the middle angular brackets, which results in a ζ-dependent expression that also includes derivatives. Integrating over ζ by parts, and employing the symmetries, we obtain the second-order response function in the form of the following spectral decomposition:

$$S^{(2)}(t_1,t_2) = \sum_{\lambda\mu} \int d\zeta\ \zeta \frac{\partial \rho_0}{\partial \zeta} e^{-\zeta\lambda t_1 - \zeta\mu t_2} \lambda \sum_{n=0}^{\infty} (-1)^n (a_n - a_{n+1}) \times \left[\left(n+s+\frac{1}{2}\right)(\zeta\mu t_2 + n) \mathcal{Q}^*_{\mu^*,n} \mathcal{Q}_{\lambda,n+1} - \left(n-s+\frac{1}{2}\right)(\zeta\mu t_2 + n - 1) \mathcal{Q}^*_{\mu^*,n+1} \mathcal{Q}_{\lambda,n}\right], \quad (67)$$

where $\mathcal{Q}_{\lambda,n}$ is given by Eq. (45). The coefficients

$$a_n = \int_{M^3} d\boldsymbol{x}\ \psi^*_n(\boldsymbol{x};s) \psi_0(\boldsymbol{x};s) \psi_n(\boldsymbol{x};s) \quad (68)$$

are purely geometrical factors that do not depend on the dynamics [1]. They are symmetric, $a_n=a_{-n}$, and can all be expressed through a_0 with the help of the recurrence relations

$$a_{n+1} = \frac{8n^2+1-4s^2}{(2n+1)^2-4s^2} a_n - \frac{(2n-1)^2-4s^2}{(2n+1)^2-4s^2} a_{n-1} \quad (69)$$

and therefore can be represented in terms of the Laplacian eigenfunction $\psi_0(\boldsymbol{x};s)$.

The spectral decomposition (67) is exact for any given κ. The diffusion coefficient κ enters the spectral decomposition via the eigenvalues as well as the coefficients $\mathcal{Q}_{\lambda,n}$. In what follows we obtain leading contributions to Eq. (67) in the noiseless limit $\kappa \to 0$.

In comparison with the linear case, noise plays a more delicate role in the spectral decomposition of the second-order response function. It provides convergence of the series over angular harmonics in the expression for the response function that does not appear in the linear case [cf. Equations (44) and (67)].

To obtain the asymptotic behavior of $\mathcal{Q}_{\lambda,n}$ for $\kappa \ll 1$ and $n \gg 1$ we first apply the method developed in Sec. VI and derive the following recurrence relation:

$$\left(\lambda - \frac{\kappa n^2}{2}\right) R_{\lambda,n} = \frac{2n+1-2s}{4} R_{\lambda,n+1} - \frac{2n-1+2s}{4} R_{\lambda,n-1}. \tag{70}$$

Specifically we have substituted $\lambda\xi_\lambda$ for $\hat{\mathcal{H}}\xi_\lambda$ in the integral representation [Eq. (39)] for $\lambda R_{\lambda,n}$ and integrated the term with the second derivative ξ''_λ by parts. Equation (70) results in another exact recurrence relation:

$$\mathcal{Q}_{\lambda,n+1} = \frac{2n-1-2s}{2n+1+2s} \mathcal{Q}_{\lambda,n-1} - \frac{4\lambda - 2\kappa n^2}{2n+1+2s} \mathcal{Q}_{\lambda,n}. \tag{71}$$

Since only even eigenfunctions contribute to the spectral decomposition, we have $\mathcal{Q}_{\lambda,-n}=\mathcal{Q}_{\lambda,n}$, which allows us to express all quantities $\mathcal{Q}_{\lambda,n}$ through $\mathcal{Q}_{\lambda,0}$ specified by Eqs. (45).

We start with the analysis of the limit $\kappa \to 0$ for fixed n. Although the dependence of $R_{\lambda,n}$ on κ is singular, the coefficients $\mathcal{Q}_{\lambda,n}$ have well-defined limits at $\kappa \to 0$ that are equal to the corresponding coefficients in the expansion (63) of the purely deterministic matrix elements $A_n(t;s)$.

This can be established in the following way. Viewing $A_n(t;s)$ as a matrix element of the evolution operator between the zeroth and nth harmonics, we use the identity $\sigma_1 = (\sigma_+ + \sigma_-)$ and the relations between the neighboring harmonics [that follow from Eqs. (5)] to express $\partial_t A_n(t;s)$ in terms of $A_{n-1}(t;s)$ and $A_{n+1}(t;s)$. Finally, the result is expanded in powers of e^{-2t}, according to Eq. (63). For the components oscillating proportionally to $e^{st/2}$ we arrive at the relation

$$Q_{\nu,n+1} = \frac{2n-1-2s}{2n+1+2s} Q_{\nu,n-1} - \frac{4\lambda}{2n+1+2s} Q_{\nu,n}, \tag{72}$$

which represents the noiseless limit of the recurrence relation (71). Combined with the already established equivalence $\lim_{\kappa\to 0} \mathcal{Q}_{\lambda_\nu,0} = Q_{\nu,0}$ for the zero term, this implies the equivalence $\lim_{\kappa\to 0} \mathcal{Q}_{\lambda_\nu,n} = Q_{\nu,n}$ for any given n.

So far equivalence has been established for the RP resonances with $\lambda_\nu = \nu - s + 1/2$. The coefficients for the other set of RP resonances (whose eigenvalues are $\lambda^*_\nu = \nu + s + 1/2$) can be easily found by employing the symmetry of the recurrence relation (71) combined with $\mathcal{Q}_{\lambda^*,0} = (\mathcal{Q}_{\lambda,0})^*$, which results in

$$\mathcal{Q}_{\lambda^*,n} = \frac{\Gamma\left(n+\frac{1}{2}-s\right)\Gamma\left(\frac{1}{2}+s\right)}{\Gamma\left(n+\frac{1}{2}+s\right)\Gamma\left(\frac{1}{2}-s\right)} (\mathcal{Q}_{\lambda,n})^* \tag{73}$$

in the limit $\kappa \to 0$. This relation establishes the equivalence $\lim_{\kappa\to 0} \mathcal{Q}_{\lambda^*_\nu,n} = \tilde{Q}_{\nu,n}$ for the other set of resonances.

We are now in a position to demonstrate that the series in angular harmonics n for any coefficient in the spectral decomposition converges for small positive κ and in the noiseless limit $\kappa \to 0$ reproduces the corresponding coefficient in the long-time asymptotic series for the purely deterministic response function $S^{(2)}(t_1,t_2)$.

Explicit expressions for the coefficients $Q_{\nu,n}$ and $\tilde{Q}_{\nu,n}$ that enter the expansion (63) for $A_n(t;s)$ become increasingly lengthy as ν grows. The coefficients are obtained by expanding the deterministic evolution matrix element in Eq. (65) in powers of e^{-2t}. The simplest expressions can be found for the lowest modes with $\nu=0$ whose energies are $\lambda = \pm s + 1/2$. For example, for the mode with $\lambda = \lambda^*_0 = s + 1/2$, the recurrence relation (72) implies $\tilde{Q}_{0,n} = (-1)^n \tilde{Q}_{0,n}$, where $\tilde{Q}_{0,n} = Q^*_{0,n}$ and $Q_{0,n}$ is specified in the right-hand side of Eq. (62) taken at $\nu=0$.

To analyze the limit $n \to \infty$ for fixed finite $\kappa \ll 1$, we notice that $R_{\lambda,n}$ are Fourier coefficients of the smooth function $\Phi_\lambda(u)$. Consequently, $\mathcal{Q}_{\lambda,n}$ decay faster than any power of n for $n \to \infty$, and indeed we find from the recurrence relations the following intermediate asymptotic of $\mathcal{Q}_{\lambda,n}$ at $1 \ll n \ll \kappa^{-1}$:

$$\mathcal{Q}_{\lambda,n} \propto (-1)^n n^{\lambda - s - 1/2} e^{-\kappa n^2/4}. \tag{74}$$

For larger n the recurrence relation implies $\mathcal{Q}_{\lambda,n}/\mathcal{Q}_{\lambda,n-1} = 1/(\kappa n)$, which also leads to a decay faster than any power law as n increases. Finally, we compare Eq. (74) with the asymptotic $n \gg 1$ form of the deterministic coefficients $Q_{\nu,n}$ to derive

$$\mathcal{Q}_{\lambda_\nu,n} = Q_{\nu,n} e^{-\kappa n^2/4}, \tag{75}$$

which is valid in the range $1 \ll n \ll \kappa^{-1}$ for small κ. Equation (75) means that noise provides a homogeneous cutoff for all RP resonances, and the series over angular harmonics for the spectral decomposition coefficients converges at $n \sim \kappa^{-1/2} \ll \kappa^{-1}$ where Eq. (75) holds.

It remains to be demonstrated that the resulting spectral decomposition in the limit $\kappa \to 0$ of vanishing noise coincides with the asymptotic long-time expansion of $S^{(2)}(t_1,t_2)$ in the deterministic case. The latter expansion has been derived in Ref. [1].

First of all, we observe that the eigenvalues of the Perron-Frobenius operator converge at $\kappa \to 0$ to the factors that appear in the expansion of $A_n(t;s)$. The same factors that appear in the spectral decomposition for the linear response have been interpreted in Ref. [14] as RP resonances. While the linear response function may be represented in the forms of a converging series, the expansion of $S^{(2)}$ appears to be more involved. If $\kappa=0$, for a pair of resonances, the series (67) over n diverges, as clearly seen from the power-law growth in Eq. (74).

In the deterministic case, the time dependence of the second-order response is determined by the converging series which contains the matrix elements $A_n(t;s)$. The expansion should be formally performed after the series summation. The long-time expansion of $S^{(2)}(t_1,t_2)$ is an asymptotic rather than a converging expansion in powers of e^{-2t_1} and e^{-2t_2}. We developed a method [1], equivalent to regrouping, which allowed the infinite series to be approximated by a sum of a finite number of terms. Due to the alternating character of the series, any smooth cutoff effective for the highest terms did not influence the result.

The noise $\kappa > 0$ actually introduces a smooth cutoff in the sum over n in Eq. (67) for a given pair of resonances λ, μ. The suppressive exponential factor is present for arbitrarily small positive κ. The series over angular harmonics is almost alternating and converging. Therefore, the sum of the series does not depend on the value of κ as long as it is small. It is

clear that another form of regularizing noise would lead to the same results.

For the second-order response function, we have demonstrated the equivalence of the spectral decomposition in the limit of vanishing noise $\kappa \to 0$ with the asymptotic expansion in the case of deterministic dynamics. In short, this follows from the equality $\lim_{\kappa\to 0} \mathcal{A}_n(t;s)=A_n(t;s)$ for the evolution operator matrix elements and the property that the asymptotic expansion of $S^{(2)}(t_1,t_2)$ originates from the expansion of $A_n(t;s)$ and the subsequent proper summation of the apparently diverging series.

VIII. DISCUSSION

In the present paper, we have studied the linear and second-order nonlinear responses of a strongly chaotic system with weak noise. The deterministic part of the model is a free particle (geodesic flow) on a compact Riemann surface of constant negative curvature. We chose the random Langevin force to be orthogonal to the particle momentum, so that the noise does not change the energy, and stochastic motion occurs on the energy shell represented by the 3D reduced phase space. The stochastic dynamics has been analyzed in terms of the Fokker-Planck operator $\hat{\mathcal{L}}=-(\kappa/2)\hat{\sigma}_z^2+\hat{L}$, where the second term stands for the deterministic component (advection), whereas the first term describes the diffusion κ being the diffusion coefficient.

In the preceding paper [1] we studied the $\kappa=0$ case that corresponds to purely deterministic dynamics. We employed strong dynamical symmetry to find an analytical solution of the problem. The Langevin noise added to the system has been chosen in a way that it does not break down the DS. Similarly to the deterministic case, where the space of reduced phase-space distributions has been decomposed into simpler components [irreducible representations of the DS group SO(2,1)] invariant with respect to the Liouville operator $\hat{L}$, the same components form invariant subspaces for our Fokker-Planck operator $\hat{\mathcal{L}}$.

Here we have considered noise as a regularization to construct spectral decompositions of the Perron-Frobenius operator and focused on the noiseless limit $\kappa \to 0$ of the stochastic dynamics. We have employed the DS to analyze the eigenvalue problem for the Fokker-Planck operator $\hat{\mathcal{L}}$ whose eigenfunctions are smooth functions for finite values of κ. Mapping the original problem onto much simpler 1D stochastic dynamics on the circle allowed for an explicit analysis of the eigenvalue problem. The relevant eigenfunctions of $\hat{\mathcal{L}}$ were obtained analytically using the WKB method where the diffusion coefficient plays the role of the Planck constant.

The fluctuation-dissipation theorem relates the linear response function to the two-point correlation function calculated earlier and interpreted as an expansion in RP resonances using the language of rigged Hilbert spaces [14]. The long-time asymptotic expression for the second-order response functions, has the form of a spectral decomposition over the same set of resonances [1]. Interpretation of the asymptotic expansion in the nonlinear case is more involved for the following reasons. The eigenmodes that correspond to the RP resonances are represented by generalized, rather than smooth, functions. In the linear case the initial smooth distribution should be decomposed into the RP modes. The signal is computed by convoluting the RP modes with the smooth dipole function. Both operations are well defined for generalized functions. In the nonlinear case the second interaction with the driving field involves applying a differential operator to a generalized function followed by projecting it onto another generalized function. The legitimacy of the latter operation is less obvious, and has been related to the cancellation of apparently dangerous terms [1].

In this paper, we have interpreted the nonlinear response in terms of RP resonances by considering the noiseless limit of the Langevin dynamics. In the case of nonzero noise, $\kappa > 0$, the spectral decomposition of the response functions of any order is conceptually straightforward. We have explicitly demonstrated that in the limit $\kappa \to 0$ the relevant eigenvalues of $\hat{\mathcal{L}}$ for motion on the compact surface of constant negative curvature converge to the RP resonances, whereas the coefficients in the spectral decompositions of the response functions converge to the coefficients of the asymptotic series in the purely deterministic $\kappa=0$ expressions derived in Ref. [1]. Thus the spectral decompositions in this limit reproduce the long-time asymptotic series for the response functions. Note that the dynamical ζ function can also be reproduced as the limit $\zeta(z)=\lim_{\kappa\to 0} Z^{-1} \det(z-\hat{\mathcal{L}})$. The spectral decomposition of the linear response is represented by a converging series. In contrast, the spectral decomposition of the second-order response function is a nonconverging asymptotic series. The rigorous convergence of the spectral decomposition is lost in the limit $\kappa \to 0$. Computation of the expansion coefficients in the nonlinear case requires a delicate summation of almost sign-alternating series whose convergence is ensured thanks to the noise.

Since all our explicit results are obtained for the particular type of noise that tends to change the momentum direction, one may wonder what happens in a general case. In other words, how is the regularization efficiency affected by the properties of infinitesimal noise? The answer requires separate study, which is beyond the scope of this paper. Nevertheless, here we present a simple estimate for the case of general homogeneous white noise. The corresponding term in the Fokker-Planck operator is given by $(1/2)h_{ij}\hat{\sigma}_i \otimes \hat{\sigma}_j$. Homogeneous noise is described by a constant matrix h_{ij}, which determines correlations of random forces [cf. Eqs. (7)–(9)]. This type of noise preserves the dynamical symmetry of the system and allows the approach developed in this paper to be applied. In the representation on the circle $0 < u \le 2\pi$, the effective diffusion operator has the form $\kappa h(u) d^2/du^2$ [see Eq. (D3) of Ref. [1]]. In the limit $\kappa \to 0$ our results, derived for $h(u)=1/2$, should remain valid if $h(0) \neq 0$ and $h(\pi) \neq 0$. Otherwise, the eigenfunctions of the Fokker-Planck operator or its adjoint become essentially different from those found in the paper, since diffusion vanishes in the relevant fixed points. Therefore, the infinitesimal noise with $h(0)=0$ or $h(\pi)=0$ may be insufficient to regularize the spectral decompositions. According to the angular representation of the stable and unstable vector fields ($\sigma_2-\sigma_z$ and $\sigma_2+\sigma_z$, respectively), random forces in both directions are

necessary for efficient regularization. This requirement can also be seen in the physical picture: "physical" relaxation of the δ-functional distribution (which corresponds to a single trajectory) occurs only in the presence of diffusion in all stable and unstable directions.

Applications of our results are not limited to interpretations of purely deterministic quantities. Irreversibility that shows itself in the decaying correlations appears only when the deterministic chaotic dynamics is regularized by some kind of coarse graining. Full physical mixing always requires some diffusion mechanism. The difference between stable and chaotic deterministic dynamics is that the diffusion-induced physical mixing is much more efficient in a chaotic system. We consider a small fraction of the phase space that represent the initial conditions. As the ball of initial conditions is stretched and folded back, the shape becomes elongated along unstable directions and contracted along stable ones, while the phase-space volume remains constant. The diffusion time scales as a square of the blurring size. Therefore, a purely diffusive relaxation in the stable system occurs on the time scale of $\tau_{\mathrm{reg}} \sim l^2/\kappa$. In a chaotic system the length scale of the density inhomogeneity decreases with time exponentially. Consider, for example, a chaotic Hamiltonian system with two degrees of freedom. In the 3D reduced phase space, a ball of size a, corresponding to initial conditions, becomes a fettuccinelike shape after some time. The fettucine length grows as $ae^{\lambda t}$. The size of the system l induces the folding of the unstable manifold. We estimate the number of folds as $\sim ae^{\lambda t}/l$. Then the characteristic physical mixing time $\tau_c \sim \lambda^{-1} \ln[l^3 a^{-2} (\lambda/\kappa)^{1/2}]$ in a chaotic system with weak diffusion is very short compared to the stable case. Note that the physical mixing time τ_c can be measured in photon echo experiments, where it represents the characteristic time scale of the photon echo decay as a function of the delay between the exciting and the dynamics-reversing pulses.

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APPENDIX: EIGENMODES OF THE FOKKER-PLANCK OPERATOR

In this appendix, we calculate relevant eigenmodes of the Fokker-Planck operator in a given representation of $G \cong \mathrm{SO}(2,1)$ labeled by s. They are represented by symmetric solutions of the Schrödinger equation on a circle with the effective Hamiltonian (24):

$$-\frac{\kappa}{2}\xi_\lambda''(u) + \left(\frac{\sin^2 u}{2\kappa} - s\cos u\right)\xi_\lambda(u) = \lambda\xi_\lambda(u). \quad \text{(A1)}$$

The Hamiltonian is self-adjoint with respect to a symmetric scalar product.

For our purposes we focus on the limit of weak noise, $\kappa \ll 1$. The low-lying states with $|\lambda| \ll \kappa^{-1}$ are concentrated near the the potential minima, in the vicinity of $u=0$ and $u=\pi$. The principal approximation for the solutions of Eq. (A1) in the regions $|u| \ll 1$ or $|u-\pi| \ll 1$ can be found by retaining dominant terms in the Taylor expansion of the potential.

At $|u| \ll 1$ we have an approximate equation

$$-\frac{\kappa}{2}\frac{d^2}{du^2}\xi_\lambda(u) + \frac{u^2}{2\kappa}\xi_\lambda(u) = (\lambda + s)\xi_\lambda(u) \quad \text{(A2)}$$

that reproduces the well-known Schrödinger equation for a linear harmonic oscillator. One can easily identify the characteristic scales $\sqrt{\kappa/\kappa}=1$ and $(\kappa\kappa)^{1/4}=\sqrt{\kappa}$ of the energy and length, respectively. A general solution of Eq. (A2) can be represented in terms of the Hermite functions as

$$\xi(u) = A_0 e^{-u^2/2\kappa} H_\nu\left(\frac{u}{\sqrt{\kappa}}\right) + B_0 e^{u^2/2\kappa} H_{-\nu-1}\left(i\frac{u}{\sqrt{\kappa}}\right), \quad \text{(A3)}$$

with A_0 and B_0 being complex constants. The parameter ν is related to the energy λ by

$$\nu = \lambda - \frac{1}{2} + s. \quad \text{(A4)}$$

In contrast to the case of a harmonic oscillator, the general solution (A3) contains both decaying and growing waves. Their relative amplitude can be determined only by solving the equation on the whole circle. The functions $H_\nu(z)$ and $e^{z^2}H_{-\nu-1}(iz)$ are linearly independent for any parameter ν and can be simply related to confluent hyperbolic functions or parabolic cylinder functions [23], the latter being solutions of the original Schrödinger equation with the harmonic potential. We prefer to deal with the Hermite functions rather than with other special functions since the former reproduce the Hermite polynomials in the case of integer order.

The general solution of Eq. (A2) near $u=\pi$ has a form similar to Eq. (A3):

$$\xi(u) = A_1 e^{-(u-\pi)^2/2\kappa} H_{\nu_1}\left(\frac{u-\pi}{\sqrt{\kappa}}\right) + B_1 e^{(u-\pi)^2/2\kappa} H_{-\nu_1-1}\left(i\frac{u-\pi}{\sqrt{\kappa}}\right), \quad \text{(A5)}$$

where

$$\nu_1 = \nu - 2s = \lambda - \frac{1}{2} - s, \quad \text{(A6)}$$

and A_1 and B_1 are two complex constants.

We begin with the construction of a symmetric solution near $u=\pi$. The requirement that the solution is symmetric with respect to $u=\pi$ combined with the relations between Hermite functions leads to an identity

$$\frac{B_1}{A_1} = i\frac{2^{\nu_1}\sqrt{\pi}}{\Gamma(-\nu_1)\cos\frac{\pi\nu_1}{2}}, \quad \text{(A7)}$$

and after some transformations we arrive at the explicit form of a symmetric solution for (A5):

$$\xi(u) = A_1 \frac{2^{\nu_1}\Gamma(\nu_1+1)}{\sqrt{\pi}} e^{(u-\pi)^2/2\kappa} e^{-i\pi\nu_1/2} \left[H_{-\nu_1-1}\left(i\frac{u-\pi}{\sqrt{\kappa}}\right) + H_{-\nu_1-1}\left(-i\frac{u-\pi}{\sqrt{\kappa}}\right)\right]. \quad \text{(A8)}$$

Therefore, in the region $|u-\pi| \ll 1$, the symmetric solution is determined by two unknown parameters $\nu \equiv \nu_1 + 2s$ and A_1; both can assume complex values.

The harmonic approximation for the potential can be used if $|u-\pi| \ll 1$, hence it is the region where the solution given by Eqs. (A5) and (A7) is valid.

A general WKB solution of Eq. (A1) under the barrier and around $u=\pi/2$ is represented by a superposition of two waves:

$$\xi(u) = \frac{A_2}{\sqrt{p(u)}} \exp[-S(u)] + \frac{B_2}{\sqrt{p(u)}} \exp[S(u)], \quad \text{(A9)}$$

where $p(u)$ and $S(u)$ are defined by

$$p(u) = \sqrt{\sin^2 u - 2\kappa s \cos u - 2\kappa\lambda}, \quad \text{(A10)}$$

$$S(u) = \frac{1}{\kappa}\int_{\pi/2}^{u} dw\, p(w). \quad \text{(A11)}$$

The semiclassical expansion over κ for the wave function phase can be employed when the wavelength $\kappa/p(u)$ does not change too fast:

$$\left| \frac{d}{du}\left(\frac{\kappa}{p(u)}\right)\right| \ll 1. \quad \text{(A12)}$$

This holds when we are not too close to the classical turning points where $p(u)$ turns to zero. A small value of κ ensures that the classical turning points lie close to $u=0$ or $u=\pi$. The WKB solutions under the barriers are valid at least for $|u| \gg \sqrt{2\kappa(\lambda+s)}$ and $|u-\pi| \gg \sqrt{2\kappa(\lambda-s)}$. If $|\lambda|, |s| \sim 1$, the validity of the semiclassical approximation (A9) is limited to the regions

$$|u|, |u-\pi| \gg \sqrt{\kappa}. \quad \text{(A13)}$$

For $\kappa \ll 1$ and $|\sin u| \gg \sqrt{\kappa}$ [i.e., in the regions determined by the same inequalities (A13)], we arrive at a simplified expression for $S(u)$:

$$S(u) = \frac{1}{\kappa}\int_{\pi/2}^{u} dw \left(\sin w - \kappa s \frac{\cos u}{\sin u} - \frac{\kappa\lambda}{\sin u}\right)$$

$$= -\frac{\cos u}{\kappa} - s \ln \sin u - \lambda \ln \tan\frac{u}{2}. \quad \text{(A14)}$$

In the preexponential factor of Eq. (A9) it is sufficient to approximate $p(u) = \sin u$.

We can see that the wave function consists of two components that exponentially grow in opposite directions under the barrier. A crude estimate for the magnitudes of the two components is presented in Fig. 1. Only the dominant part $\sin^2 u/(2\kappa)$ of the potential contributes to the exponential dependence of the barrier transparency on κ.

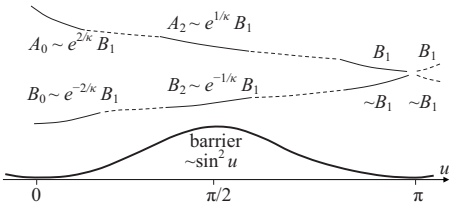

FIG. 1. Schematic picture of the amplitudes that result from the tunneling through the potential barrier.

We further notice that for u close to π [actually in the interval $\kappa^{1/2} \ll (\pi-u) \ll \kappa^{1/4} \ll 1$] we can use the following approximation of Eq. (A14):

$$S(u) = \frac{1}{\kappa} - \frac{(u-\pi)^2}{2\kappa} + (\lambda - s)\ln(\pi - u) - \lambda \ln 2,$$

where the neglected terms are small compared to unity. This is the interval where the WKB solution (A9) under the barrier and the solution (A8) in the harmonic well both represent a good approximation and can be matched. Using the asymptotic expressions for the Hermite functions, we compare the components that grow with an increase of $|\pi-u|$ and obtain

$$B_1 = A_2 e^{-1/\kappa} 2^{\nu_1+\lambda+1} \kappa^{-(\nu_1+1)/2} e^{i\pi(\nu_1+1)/2}. \quad \text{(A15)}$$

The value of A_1 is then found from Eq. (A7).

First we assume that the ratio B_1/A_1 given by Eq. (A7) does not contain an exponentially small term $e^{-1/\kappa}$. Matching the amplitude of the component decreasing as u deviates from π, we find an estimate

$$B_2 \sim B_1 e^{-1/\kappa}. \quad \text{(A16)}$$

The procedure of solving the Schrödinger equation near $u=0$ is completely similar. In the region $\sqrt{\kappa} \ll |u| \ll 1$ both approximate solutions (A3) and (A9) are valid and should be matched. To that end we expand $S(u)$ in the region $|u| \ll \kappa^{1/4} \ll 1$,

$$S_2(u) = -\frac{1}{\kappa} + \frac{u^2}{2\kappa} - (\lambda + s)\ln u + \lambda \ln 2,$$

neglecting the terms that are small compared to unity, and also employ the asymptotic forms of the Hermite functions in the overlap region. When u approaches 0, the second term in Eq. (A9) decreases and hence matches the second term in Eq. (A3),

$$B_0 \sim B_2 e^{-1/\kappa} \sim B_1 e^{-2/\kappa}. \quad \text{(A17)}$$

Equations (A15)–(A19) imply that the second component in Eq. (A3) scales $B_0 \propto e^{-4/\kappa} A_0$ near $u=0$. Its presence is related to the nonresonant tunneling along the circle from the potential minimum into itself. The argument in the suppressive exponential factor is evaluated as

$$\frac{1}{\kappa}\int_0^{2\pi} du\, p(u) = \frac{1}{\kappa}\int_0^{2\pi} du |\sin u| = \frac{4}{\kappa}. \quad \text{(A18)}$$

The tunneling cannot be resonant, since the two wells near $u=0$ and $u=\pi$ are "offset" by the imaginary number $2s$.

Finally, the matching of components with A_2 and A_0 yields

$$A_2 = A_0 e^{-1/\kappa} 2^{\nu+\lambda} \kappa^{-\nu/2}. \tag{A19}$$

Performing the same matching procedure in the region $-\pi \le u < 0$, as was done for $0 < u \le \pi$, we find the following even solution valid for $0 < |u| \ll 1$:

$$\xi(u) = A_0 e^{-u^2/2\kappa} H_\nu\left(\frac{|u|}{\sqrt{\kappa}}\right). \tag{A20}$$

For arbitrary ν the function may have a cusp at $u=0$. The solution is smooth if ν is a nonnegative even integer. Then the Hermite functions H_ν reduce to Hermite polynomials of even order. Therefore, from Eq. (A4) we obtain the eigenvalues $\lambda_\nu = \nu + (1-2s)/2$ with $\nu = 0, 2, \ldots$, for the states concentrated near $u=0$. Exponentially small corrections proportional to $e^{-4/\kappa}$ due to the tunneling can be neglected. The other type of corrections, in powers of κ, that originate from the deviation of trigonometric functions from their harmonic approximations, cannot be dismissed. The method developed in Sec. IV allows us to avoid calculating the power law corrections completely.

The other set of eigenstates can be found by considering the complex conjugation of the Hamiltonian, which corresponds to a change of sign of s, or to the shift $u \to u + \pi$. The resulting quantization condition is $\nu_1 = 2k$ with $k = 0, 1, 2, \ldots$. These eigenstates have energies $\lambda = 2k + (1+2s)/2$ and are concentrated near $u=\pi$. The corresponding eigenfunctions are easily derived from the ones specified above.

The eigenfunctions of the Hamiltonian (24) found above differ significantly from those of the linear harmonic oscillator only far from the interval $|u| \lesssim \kappa^{1/4}$. Outside the interval the exponentially decaying functions take negligibly small values. Consequently, the normalization factor is

$$A_{0,\nu} = (\pi\kappa)^{-1/4} (2^\nu \nu!)^{-1/2}. \tag{A21}$$

The phase of the normalized eigenfunctions of Hermitian operators can be chosen arbitrarily. For our Hamiltonian and scalar product (25), this reduces to freedom in the sign choice.

For illustrative purposes, we conclude by calculating the correction to the energy $\lambda_\nu = \nu - s + 1/2$ of the eigenstate (note that the following result is not used anywhere):

$$\lambda_\nu^{(1)} = \int_{-\pi}^{\pi} du \left(\frac{su^2}{2} - \frac{u^4}{6\kappa}\right) \xi_{\lambda_\nu}^2(u) \tag{A22}$$

$$= \frac{\kappa}{8}[2s(1+2\nu) - (1+2\nu+2\nu^2)]. \tag{A23}$$

One of the ways to derive Eq. (A22) is to consider the first neglected term in the WKB expansion under the barrier. The WKB wave function contains the state energy. One can simply require that the wave function under the barrier does not contain logarithmic terms due to their absence in the approximate solution near $u=0$.

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